Similarity Type of Solutions of the Turbulent Boundary Layer Equations for Momentum and Energy

Part I: An Analysis of Some Existing Solutions

A. S. TELLES and A. E. DUKLER

University of Houston, Houston, Texas

The law of the wall, the law of the wake, and the velocity defect law have been proposed as similarity types of solutions of the turbulent boundary layer momentum equation. Spalding proposed an approximation of the same type as the solution of the energy equation. This study shows these laws to be derivable from a small-perturbation type of solution of the general boundary layer equations for turbulent flow. The limiting conditions where each can be expected to be valid are clearly demonstrated. In particular, the law of the wall is shown to be the zeroth-order solution of the momentum equations when the solution is expressed in the form of a small-perturbation expansion. The first-order perturbation is shown to generate a relationship similar to Coles' law of the wake. Likewise, Spalding's approximation is shown to be the zeroth-order small-perturbation type of solution of the energy equation.

The energy equation for a turbulent boundary layer requires for its solution values of the local velocity, u and v, as well as an expression for the eddy thermal conductivity. As most models for eddy conductivity aro based on an analogy between heat and momentum transfer, the eddy thermal conductivity depends (as does the eddy viscosity) on the local velocity and/or the velocity gradients. Values for the local eddy conductivity as well as for local velocities thus depend on a solution of the momentum equation for the turbulent boundary layer. When the physical properties are considered independently of temperature, the momentum equation is uncoupled from the energy equation and, in principle, the solution of the energy equation can be accomplished after the momentum equation has been solved. Much attention has thus been given to the solution of this momentum equation for the turbulent boundary layer.

Solutions of these partial differential equations for momentum transfer have been impossible, due to their nonlinear character. Both experimental and analytical attacks on the problem of the turbulent boundary layer have been reviewed thoroughly by Clauser (*2*) and Hinze (*6*). Most methods have in common the use of a similarity variable to reduce the partial differential equations involved to ordinary ones. It is assumed that the new variable for velocity depends only on a single-position variable which characterizes both the distance from the wall and the distance in the flow direction [as Blasius (*1*) assumed for laminar flow]. In this way the *law of the wall* proposes that in the region near the wall, when velocity and distance from the wall are expressed in terms of the variables u^+ and y^+, the dimensionless velocity u^+ depends only on the dimensionless distance from the wall and is independent of the coordinate measured in the direction of flow.

$$u^+ = u^+(y^+)$$

This is equivalent to stating that at all positions along the boundary layer in the direction of flow the velocity profiles expressed in these coordinates would be identical. In a like manner a *velocity defect law* has been in use for the fully developed turbulent region of the boundary layer. This law suggests that the decrease in velocity from the free stream velocity is a unique function of the relative position from the wall.

$$\frac{u_\infty - u}{u_*} = u_D = u_D(\xi)$$

Each of these laws is asymptotic in character and applies only over limited regions. Much effort has been directed toward finding corrections and toward evolving methods of matching solutions from these two laws (2).

Additional assumptions are required in order to solve the energy equations once these approximate solutions to the momentum equation have been obtained. In many cases one or more of the following assumptions are used:

a. Thermal-boundary-layer thickness is small, permitting the use of the law of the wall.
b. Heat flux is constant and thus independent of position from the wall.
c. Convective terms in the direction normal to the wall are negligible.
d. Prandtl number is unity.

Frequently it is difficult to determine all the assumptions and limitations which apply to any one method. Since exact solutions of the turbulent boundary layer equations are hardly to be expected, it becomes important to evolve a solution which is least wrong and which has the highest degree of consistency with the boundary layer equations it is intended to solve. In the first of these papers several well recognized solutions to the momentum and energy equations will be examined in this light. The second paper will present a new solution based on a Blasius type of similarity proposal which not only will be shown to be less wrong but which incorporates a method of evaluating the error and providing correction if desired.

A MATHEMATICAL STATEMENT OF THE PROBLEM

Under the assumptions of nondissipative boundary layer flow of an incompressible and barotropic fluid, the equations for conservation of mass, momentum, and energy in a turbulent boundary layer are

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

conservation of momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial}{\partial y}\left[(1+\epsilon/\nu)\frac{\partial u}{\partial y}\right] - \frac{1}{\rho}\frac{\partial P}{\partial x} \tag{2}$$

conservation of energy

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial}{\partial y}\left[\left(1+\frac{\epsilon_H}{\alpha}\right)\frac{\partial T}{\partial y}\right] \tag{3}$$

A unique solution of these equations satisfying these boundary conditions, is desired, once the form for the eddy diffusivities is prescribed.

$$\begin{array}{llll} y = 0: & u = v = 0 & y \to \infty & u \to u_\infty \\ & T = T_w & & T \to T_\infty \end{array} \tag{4}$$

The difficulty in handling these equations has led investigators to assume similarity or universality of velocity and temperature profiles which makes it unnecessary to solve the equations. The law of the wall and the defect law, as well as Coles's wake law, are examples of approximations of the momentum equation with Spalding's equation (17), an example for the energy equation. These empirical equations can be generated from Equations (1) to (4) above. At the same time it will be possible to show clearly the nature of the assumptions implied. Furthermore, Coles's empirical *law of the wake* (4) will be shown to have a basis in theory, although not in the exact form he proposed.

The Law of the Wall

The law of the wall is based on the premise that velocity profiles are independent of position along the plate when the velocity is measured in terms of its scale—u_*, the friction velocity—and the distance from the wall is measured in terms of its scale—ν/u_*—or

$$u^+ = \frac{u}{u_*} = u^+(y^+) \tag{5}$$

where

$$y^+ = \frac{u_* y}{\nu} \tag{6}$$

In the general case, since local velocity depends on both x and y, u^+ will depend on y^+ and also on a quantity representing the x dimension, say x^+ or

$$u^+ = u^+(y^+, x^+) \tag{7}$$

In order to determine the condition under which the law of the wall [Equation (5)] is valid, u^+ and y^+ are introduced into the momentum equations as a transformation of variables by means of Equation (7). The result can be written as

$$\frac{\partial}{\partial y^+}\left[\epsilon^+\frac{\partial u^+}{\partial y^+}\right] = \frac{\nu}{u_*^3\rho}\frac{\partial P}{\partial x} + u^{+2}\left\{\frac{\nu}{u_*^2}\frac{du_*}{dx} + \frac{\partial}{\partial y^+}\left[\frac{(\partial\psi^+/\partial x^+)_{y^+}}{(\partial\psi^+/\partial y^+)_{x^+}}\right]\right\} \tag{8}$$

The new terms in the equation are defined below:

$$\epsilon^+ = 1 + \epsilon/\nu \tag{9a}$$

$$\psi^+ = \frac{\psi}{\nu} = \int_0^{y^+} u^+ dy^+ \tag{9b}$$

$$x^+ = \int_0^x \frac{u_*}{\nu}dx \tag{9c}$$

The stream function term in square brackets, which has important physical significance, will be discussed later. Equation (8) can be condensed to this convenient form:

$$\frac{\partial}{\partial y^+}\left[\epsilon^+\frac{\partial u^+}{\partial y^+}\right] = Z(u^{+2} + p^+) \tag{10}$$

where

$$Z = \frac{\nu}{u_*^2}\frac{du_*}{dx} + \frac{\partial\Delta}{\partial y^+} \tag{11a}$$

$$p^+ = \frac{\nu}{Zu_*^3\rho}\frac{\partial P}{\partial x} \tag{11b}$$

$$\Delta = \frac{(\partial\psi^+/\partial x^+)_{y^+}}{(\partial\psi^+/\partial x^+)_{x^+}} \quad (11c)$$

If Z can be shown to be a small number and dependent only on x^+, the solution to Equation (10) can be approached by a small-perturbation expansion of the type

$$u^+ = u_0{}^+ + Zu_1{}^+ + Z^2u_2{}^+ + \ldots = \sum_{n=0}^{\infty} Z^n u_n{}^+ \quad (12)$$

where $u_0{}^+$, $u_1{}^+$, $u_2{}^+$ ... are successively the zeroth, first-, second-, and higher-order perturbation solutions. The first term making up Z in Equation (11a) is indeed a small number. For example, for zero pressure gradient it is readily shown to be of the order of $Re_x{}^{-9/10}$. Furthermore, it depends only on x^+. Under the conditions where Δ varies linearly with y^+ or is constant, the second term in Equation (11a) is independent of y^+, and the conditions required for the use of the small-perturbation type of expansion are thus satisfied.

Substituting for u^+ in Equation (10), by the expansion of Equation (12) and the collection of terms of like power in Z, produces the series of differential equations which are solved to find $u_0{}^+$, $u_1{}^+$, $u_2{}^+$, etc. The first few equations are

$$\frac{\delta}{\delta y^+}\left[\epsilon_0{}^+ \frac{\partial u_0{}^+}{\delta y^+}\right] = 0 \quad (13)$$

$$\frac{\delta}{\delta y^+}\left[\epsilon_1{}^+ \frac{\partial u_1{}^+}{\delta y^+}\right] = u_0{}^{+2} + p^+ \quad (14)$$

$$\frac{\delta}{\delta y^+}\left[\epsilon_2{}^+ \frac{\partial u_2{}^+}{\delta y^+}\right] = 2u_0{}^+u_1{}^+, \text{etc.} \quad (15)$$

Because ϵ^+ depends, in general, on u^+ and y^+, its form will differ in each of the equations; this is designated above by $\epsilon_0{}^+$ for its form in the zeroth-order equation, by $\epsilon_1{}^+$ for the first order equation, etc. For the zeroth order of perturbation, solution of Equation (13) with the condition that at $y^+ = 0$, $du^+/dy^+ = 1.0$ and $\epsilon_0{}^+ = 1.0$ gives

$$u_0{}^+ = \int_0^{y^+} \frac{dy^+}{(1 + \epsilon/\nu)} \quad (16)$$

If ϵ^+ is a unique function of y^+, then $u_0{}^+$ depends only on y^+ and Equation (17) becomes a law of the wall. In the turbulent zone $\epsilon/\nu >> 1.0$ and with the Prandtl assumption that ϵ/ν varies linearly with y^+, the integration yields the familiar logarithmic form of the law of the wall:

$$u_0{}^+ = A + B \ln y^+ \quad (17)$$

It is now clear that the law of the wall may be considered the zeroth-order term in a small-perturbation type of solution to the boundary layer equations expressed in $u^+ - y^+$ coordinates. Additional assumptions include

a. The function Δ varies linearly with y^+.
b. ϵ/ν is a unique function of y^+.

The Law of the Wake

Coles (4) proposed a purely empirical correction to the law of the wall to account for the large observed deviations between Equation (18) and the experiment at high values of y^+,

$$u^+ = A + B \ln y^+ + K_1 W(\xi) \quad (18)$$

In this equation $W(\xi)$ is a function only of relative position, $\xi = y/\delta$. On the basis of experimental evidence, Coles considered this functional relationship to be universal, and he presented numerical values of W for various values of ξ. He proposed that the constant, K_1, be evaluated from the condition existing at $y = \delta$;

$$\frac{1}{\sqrt{C_f/2}} = A + B \ln \frac{u_*\delta}{\nu} + 2K_1 \quad (19)$$

While Coles arrived at his wake law by empirical means, it is possible to show that by considering the first perturbation term, one can derive a similar relationship directly from a perturbation type of solution of the boundary layer equations. This can be accomplished in the following manner.

Clauser (2), Hinze (6), and others have shown that the eddy viscosity varies little in the outer region of a boundary layer where the shear stress changes rapidly but displays most of its variation in the region of the wall where the stress changes only slightly. The analysis of Hinze using data of several investigators suggests that ϵ^+ becomes essentially constant at $\xi \simeq 0.16$. This location for constant ϵ^+ is assumed the same for boundary layers at other Reynolds numbers and pressure gradients. The validity of this assumption, based on phenomenological observation, will be demonstrated *a posteriori*. The differential equations for the first two terms in the expansion for u^+ are now developed and solved for each of these two regions.

Region I. $\xi \leqq 0.16$. The Prandtl equation for ϵ^+ applies in this region:

$$\epsilon^+ = \frac{\epsilon}{\nu} = k^2 y^{+2} \frac{du^+}{dy^+} \quad (20)$$

Substituting for ϵ^+ from Equation (20) and for u^+ with Equation (12) into (10) and collecting terms of like powers in Z gives these two equations:

$$\frac{d}{dy^+}\left[k^2 y^{+2}\left(\frac{\partial u_0{}^+}{\partial y^+}\right)^2\right] = 0 \quad (21a)$$

$$\frac{d}{dy^+}\left[2k^2 y^{+2} \frac{du_0{}^+}{dy^+}\frac{du_1{}^+}{dy^+}\right] = (u_0{}^+)^2 + p^+ \quad (21b)$$

Sequential solution of these equations with $y^+ = 30$ used as a lower bound in Equation (21b), at which point $u_1{}^+ = 0$, $(du_1{}^+/dy^+) = 0$, gives

$$u_0{}^+ = A + B \ln y^+ \quad (22a)$$

$$u_1{}^+ = a_0 + a_1 y^+ + a_2 y^+ \ln y^+ + a_3 y^+ (\ln y^+)^2 + a_4 \ln y^+ \quad (22b)$$

Solving these equations is not difficult but is very involved algebraically. The coefficients a_1–a_4 are complex algebraic functions of the constants A and B of Equation (22a) as well as of p^+, Z, and $y_m{}^+$. These will be transformed into a more useful form and discussed below.

After substitution of these expressions for $u_0{}^+$ and $u_1{}^+$ into Equation (12), the resulting equation can be compared with Equation (18), proposed by Coles. It becomes clear that Coles's wake law for this region is

$$K_1 W(\xi) = Z u_1{}^+ \quad (23)$$

Recognizing that

$$y^+ = y_m{}^+ \xi \quad (24)$$

$$y_m{}^+ = \frac{u_*\delta}{\nu} \quad (25)$$

permits the transformation of (23) into the following form by use of (22b):

$$W(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2 \ln \xi + \alpha_3 \xi \ln \xi + \alpha_4 \xi (\ln\xi)^2 \quad (26)$$

The equations for α_0 to α_4 are to be found in Table 1.

TABLE 1. COEFFICIENTS FOR THE WAKE FUNCTIONS, EQUATIONS (26) AND (31)

$$\alpha_0 = -\frac{K_2}{y_M{}^+ K_1}\left[15B\left(\frac{K_3}{K_2}+J_1\right)\ln y_M{}^+ + 15B(\ln 30 - 1)\frac{K_3}{K_2} + J_0\right]$$

$$\alpha_1 = \frac{B}{2}\frac{K_3}{K_1} + \frac{K_2}{K_1}\left[3B^3 - 2AB^2 + A^2B + B^2(A-2B)\ln y_M{}^+ + \frac{B^3}{2}(\ln y_M{}^+)^2\right]$$

$$\alpha_2 = -\frac{15B}{y_M{}^+}\frac{K_2}{K_1}\left(\frac{K_3}{K_2}+J_1\right)$$

$$\alpha_3 = \frac{BK_2}{2K_1}\left[2AB - 4B^2 + 2B^2\ln y_M{}^+\right]$$

$$\alpha_4 = \frac{B^3K_2}{2K_1}$$

$$\beta_0 = \frac{15\,BK_3}{y_M{}^+ K_1}\left[\frac{0.16\,y_M{}^+}{30} - \ln Y_M{}^+ - \ln 0.16 - \ln 30 + 1\right] = \frac{K_2}{K_1}J_2 + \frac{1}{K_1}\left[U_{0c}{}^+ - B(\ln y_M{}^+ - 1) - A\right]$$

$$\beta_1 = -\frac{B}{K_1}$$

$$\beta_2 = \frac{B}{0.16\,K_1} + \frac{BK_2}{K_1}\left[-\frac{K_3}{K_2}\left(1+\frac{30}{0.16\,y_M{}^+}\right) + J_3 - 2(\overset{+}{U_{0c}})^2 + 2B\overset{+}{U_{0c}} - \frac{2}{3}B^2\right]$$

$$\beta_3 = \frac{B}{2(0.16)}\frac{K_2}{K_1}\left[\frac{K_3}{K_2} + \overset{+}{U_{0c}}{}^2 - 2B\overset{+}{U_{0c}} + B^2\right]$$

$$\beta_4 = \frac{B^2}{3(0.16)^2}\frac{K_2}{K_1}[\overset{+}{U_{0c}} - B]$$

$$\beta_5 = \frac{B^3}{12(0.16)^3}\frac{K_2}{K_1}$$

$$J_0 = 15B\,[4AB - A^2 - 3B^2 - 2 + \ln 30\,(2B^2 - 2AB - B^2\ln 30 + 2 + J_1)]$$

$$J_1 = A^2 + B^2[(\ln 30)^2 - 2\ln 30 + 2] + 2AB(\ln 30 - 1)$$

$$J_2 = \frac{0.16B}{2}\left[J_4 + \frac{2}{0.16}((\overset{+}{U_{0c}})^2 - 2B\overset{+}{U_{0c}} + B^2) + \frac{2}{3}B(\overset{+}{U_{0c}} - \frac{3}{4}B) - 2(\overset{+}{U_{0c}})^2 + 2B\overset{+}{U_{0c}} - \frac{2}{3}B^2\right]$$

$$J_3 = \ln y_M{}^+\,[2(AB - B^2) + 2B^2\ln 0.16] + B^2(\ln y_M{}^+)^2 + \frac{30}{0.16\,y_M{}^+}J_1 + (A-B)^2 + 2(AB - B^2)\ln(0.16) + B^2(\ln 0.16)^2$$

$$K_2 = Zy_M{}^+$$

$$K_3 = Zy_M{}^+\,p^+ = \frac{\delta}{\rho u_*{}^2}\frac{dP}{dx}$$

Region II: $\xi \geqq 0.16$. Here the eddy viscosity is constant at the value it assumes at $\xi = 0.16$ and is designated by $\epsilon_c{}^+$. The differential equations for the first two terms in the perturbation expansion equation for u^+ in this region are

$$\frac{d}{dy^+}\left[\epsilon_c{}^+\frac{du_0{}^+}{dy^+}\right] = 0 \qquad (27a)$$

$$\frac{d}{dy^+}\left[\epsilon_c{}^+\frac{du_1{}^+}{dy^+}\right] = (u_0{}^+)^2 + p^+ \qquad (27b)$$

Equations (27*a*) and (27*b*) are integrated with the conditions

$$@\quad \xi = 0.16,\quad y^+ = y_0{}^+ = 0.16\,y_m{}^+$$
$$u_0{}^+ = u_{0c}{}^+;\quad u_1{}^+ = u_{1c}{}^+ \qquad (28)$$
$$(du_0{}^+/dy^+) = (du_0{}^+/dy^+)_c$$
$$(du_1{}^+/dy^+) = (du_1{}^+/dy^+)_c$$

These lower bounds can, of course, be found from Equations (22*a*) and (22*b*) by setting $y^+ = 0.16y_m{}^+$. The solutions for Region II are

$$u_0{}^+ = u_{0c}{}^+ + \frac{y^+ - y_c{}^+}{\epsilon_c{}^+} \qquad (29a)$$

$$u_1{}^+ = b_0 + b_1y^+ + b_2(y^+)^2 + b_3(y^+)^3 + u_{1c}{}^+ \qquad (29b)$$

For this region a comparison of Equations (12) and (18) shows that Coles's correction to the wall law is

$$K_1W(\xi) = u_0{}^+ + Zu_1{}^+ - A - B\ln y^+ \qquad (30)$$

Using Equation (29) permits an equation for $W(\xi)$ to be developed of this form:

$$W(\xi) = \beta_0 + \beta_1\ln\xi + \beta_2\xi + \beta_3\xi^2 + \beta_4\xi^3 + \beta_5\xi^4 \qquad (31)$$

The complex equations for the coefficients, β_0 to β_5, are found in Table 1. The variables, Z and p^+, have been regrouped to more convenient forms, and new variables appear in Table 1 as

$$K_2 = Zy_m{}^+ \qquad (32)$$

$$K_3 = Zp^+y_m{}^+ = \frac{\delta}{\rho u_*{}^2}\frac{dP}{dx} \qquad (33)$$

An examination of Table 1 makes it clear that, once values for the constants A and B of Equation (22*a*) have been selected, all coefficients depend only on the parameters K_1, K_2, K_3, and $y_m{}^+$. Thus the wake functions for each of the regions are similarly dependent:

$$W(\xi) = \phi(K_1, K_2, K_3, y_m{}^+)$$

Two of these parameters can be eliminated from two additional relationships provided by the normalizing conditions used by Coles.

$$W(\xi) = 2\ @\ \xi = 1.0 \qquad (34a)$$

$$\int_0^1 W(\xi)d\xi = 1.0 \qquad (34b)$$

In this way K_2 was eliminated and a relationship developed between K_1, K_3, and $y_m{}^+$. Based on the values of A and B recommended by Coles ($A = 2.5$, $B = 5.1$), the numerical results are shown in Table 2. The insensitivity of the relation between K_1 and K_3 to the value of $y_m{}^+$ should be noted. The data are well represented by a single equation

$$K_1 = 1.47 + 0.570K_3 \qquad (35)$$

over the range $1{,}000 < y_m{}^+ < 10{,}000$, for which numerical values were calculated.

It is now possible to calculate the wake functions for each of the regions by means of Equations (26) and (31), once the values of K_3 and $y_m{}^+$ are specified. The results of a series of such calculations appear in Figure 1 for values of K_3 varying from 0 to 25 with $y_m{}^+$ from 1,000 to 10,000. Thus the quantity that Coles calls the "wake law" is seen to depend parametrically on K_3, the parameter

Table 2. K_1 for Various Values of the Parameter, y_M^+ and K_3

y_M^+	Values of K_1 at K_3 = 0	5	10	15	20	25
1,000	1.58	4.33	7.06	9.80	12.54	15.28
2,000	1.52	4.37	7.22	10.06	12.91	15.76
3,000	1.49	4.36	7.23	10.10	12.97	15.84
5,000	1.46	4.32	7.19	10.08	12.92	15.78
10,000	1.42	4.24	7.05	9.87	12.68	15.50

which characterizes the pressure gradient. Rather remarkably, the $W(\xi)$ curves are only weak functions of K_3 and y_m^+ even over the wide range of these parameters covered here.

Figure 2 compares the location of the curve obtained by Coles directly from the data with the range of values presented in Figure 1 and obtained here from a perturbation type of solution of the boundary layer equations. The agreement is excellent. For a zero pressure gradient, where K_3 is zero, Equation (35) predicts a value of $K_1 = 1.47$. By observation of the data, Coles suggested that $K_1 = 1.37$. The two values are very close indeed.

It seems reasonable that a correction to the wall law should, to some degree, depend on the pressure gradient and the scaled thickness of the boundary layer itself, y_m^+. In fact, these curves, derived from the boundary layer equations themselves, are in close accord with observations. A reexamination of the zero pressure gradient data used by Coles and of the recent data of Smith (*11*) also for zero gradient clearly shows a region of negative $W(\xi)$ as ξ approaches zero and a region where $W(\xi)$ is greater than 2.0 in the region between ξ of 0.80 and 1.0. Both these observations agree with the trends shown in the theoretical curves of Figure 2.

The term

$$K_3 = \frac{\delta}{u_* \rho} \frac{dP}{dx}$$

arises naturally from this analysis. Clauser (*2*) uses the same term to characterize the equilibrium boundary layer.

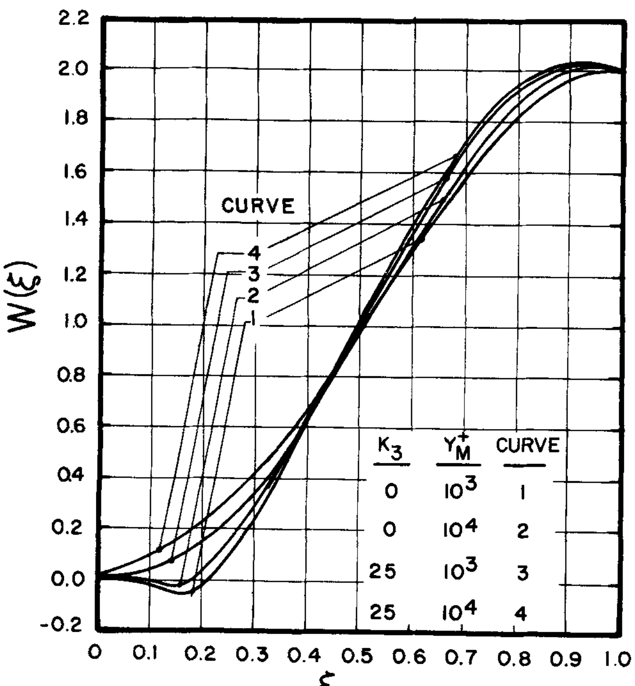

Fig. 1. The wake function predicted from the perturbation solution of boundary layer equation.

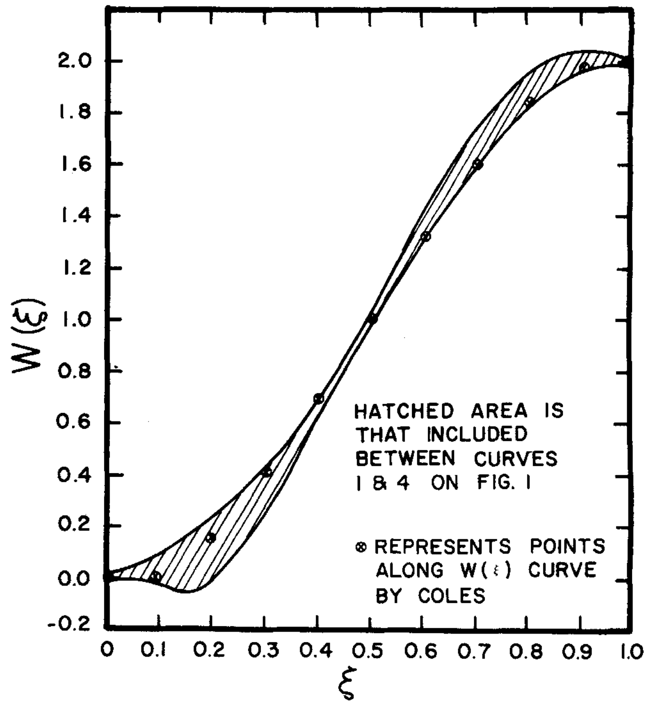

Fig. 2. Comparison with Coles' wake function.

In fact, Clauser defines an equilibrium boundary layer as one in which K_3 is a constant along the length.

It thus appears that a wake law correction to the law of the wall arrived at by Coles empirically can be developed directly from a solution of the boundary layer equations by solving the equations reformulated in a small perturbation form. The zeroth-order solution yields the law of the wall. The first perturbation yields a correction term almost identical to that of Coles's wake law. There is thus a basis for the wake law in the boundary layer equations themselves.

THE NATURE OF SIMILARITY AND THE MEANING OF THE Δ FUNCTION

In order to develop the perturbation expansion for the velocity profile we assumed that the Δ function defined in Equation (11*c*) was linearly dependent on y^+. Some remarks about the nature of similarity will remove the arbitrariness of this assumption.

If a physical quantity, F, is a function of two length coordinates g and h, similarity will exist if there exists an F scale, F_s, and a length scale, h_s, such that the ratio $F/F_s = F^+$ is a unique function of the ratio $h/h_s = h^+$. That is, when F is measured in its own scale, F_s, it is uniquely determined by h^+, the length coordinate measured in its own scale, h_s. This condition of independence of g means that $(\partial F^+/\partial g)_{h^+} = 0$. The rules of partial differentiation give

$$\left(\frac{\partial F^+}{\partial g}\right)_{h^+} = \left(\frac{\partial F^+}{\partial h}\right)_g \left(\frac{\partial h}{\partial g}\right)_{h^+} + \left(\frac{\partial F^+}{\partial g}\right)_h = 0 \tag{36}$$

Dividing this expression by $(\partial F^+/\partial h)_g$ and using the relation

$$\left(\frac{\partial F^+}{\partial g}\right)_h \left(\frac{\partial g}{\partial h}\right)_{F^+} \left(\frac{\partial h}{\partial F^+}\right)_g = -1 \tag{37}$$

yields the following equation:

$$\left(\frac{\partial h}{\partial g}\right)_{h^+} = \left(\frac{\partial h}{\partial g}\right)_{F^+} \tag{38}$$

The last equation states that for similarity to exist (that is, for F^+ to be independent of g for a given value of h^+), curves of h vs. g at constant h^+ have the same slope as those drawn at constant F^+ for all points in the field. Any deviation from parallelism can be represented by the difference in these slopes, $(dh/dg)_{h^+} - (dh/dg)_{F^+}$. The Δ function defined in Equation (11c) can be related to this difference as shown below.

Let the variable of interest be the stream function, $\Psi^+ = \Psi^+(x^+, y^+)$. Then by the rules of partial differentiation,

$$\left(\frac{\partial\psi^+}{\partial x^+}\right)_{y^+} = \left(\frac{\partial\psi^+}{\partial y}\right)_{x^+} \left(\frac{\partial y}{\partial x^+}\right)_{y^+} + \left(\frac{\partial\psi^+}{\partial x^+}\right)_{y} \tag{39}$$

In the general case the x^+ derivative is, of course, not zero. Applying an equation similar to (36) gives, after some rearrangement,

$$\Delta = \frac{\left(\frac{\partial\psi^+}{\partial x^+}\right)_{y^+}}{\left(\frac{\partial\psi^+}{\partial y^+}\right)_{x^+}} = \left(\frac{\partial y}{\partial x}\right)_{y^+} - \left(\frac{\partial y}{\partial x}\right)_{\psi^+} \tag{40}$$

The right side of this equation will be recognized as the measure of difference in parallelism between curves drawn in the boundary layer which relate x and y at constant y^+ and constant ψ^+. For complete similarity these curves are parallel, the slopes are equal at all points, and the right side of Equation (40) is zero and so is Δ. In turbulent flow the most that can be expected is near parallelism, and this can be expected only in certain regions of flow.

It is possible to estimate the value of Δ at the extremes of the boundary layer for any transformation of variables. This is done here for the $u^+ - y^+$ transformation discussed above.

As y Approaches 0

From the definition of ψ^+, Equation (9b), and the condition of zero velocity at the wall it is seen that

$$\lim_{y^+ \to 0} \left(\frac{\partial\psi^+}{\partial x^+}\right)_{y^+} = 0 \quad \text{and} \quad \lim_{y^+ \to 0} \Delta = 0 \tag{41}$$

As y Approaches δ

In this region the velocity approaches a constant which is near u_∞. Integration of Equation (9b) with this consideration gives

$$\psi^+ \cong a + \frac{u_\infty}{u_*} y^+ \tag{42}$$

$$\left(\frac{\partial\psi^+}{\partial x^+}\right)_{y^+} = \left(\frac{\partial a}{\partial x^+}\right)_{y^+} + y^+ \frac{d}{dx^+}\left(\frac{u_\infty}{u_*}\right) \tag{43}$$

where a is a quantity independent of y^+. Since $(\partial\psi^+/\partial y^+)_{x^+}$ of Equation (40) is independent of y^+, then Δ is linearly dependent on y^+. For similarity to exist, Δ must be zero; so it is clear that similarity in the $u^+ - y^+$ coordinates cannot possibly exist near the edge of the boundary layer. However, the fact that the deviation is linear in y^+ does make Z of Equation (11a) independent of y^+, and this makes it possible to use the perturbation technique to attempt to solve the boundary layer equations. Since Δ is independent of y^+ near the wall and linearly dependent near the outer edge, there must be a region (probably close to the wall) where the behavior of Δ is nonlinear with y^+, and in this region even the use of the small perturbation solution is subject to question. In the second paper of this series a transformation will be presented where Δ is shown to be negligible at both ends of the boundary and small in the intermediate region.

THE ENERGY EQUATION

Spalding (12) has indicated the utility of transforming the energy equation so that the dimensionless temperature, θ, becomes dependent on the independent variables u^+ and x^+ rather than on the usual position variables x^+ and y^+. He presents and solves a simplified form of these equations. It is again of interest to observe the exact form of the equation and the conditions under which the Spalding approximations may be valid.

Define θ as

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} = \theta(x^+, u^+) \tag{44}$$

The derivatives of θ as they appear in Equation (3) are calculated in terms of these new variables. For example,

$$\left(\frac{\partial\theta}{\partial x}\right)_y = \left(\frac{\partial\theta}{\partial x^+}\right)_{u^+} \left(\frac{\partial x^+}{\partial x}\right)_y + \left(\frac{\partial\theta}{\partial u^+}\right)_{x^+} \left(\frac{\partial u^+}{\partial x}\right)_y$$

$$= \frac{u_*}{\nu}\left[\left(\frac{\partial\theta}{\partial x^+}\right)_{u^+} + \left(\frac{\partial\theta}{\partial u^+}\right)_{x^+} \left\{\frac{\nu}{u_*^2} y^+ \frac{du_*}{dx}\left(\frac{\partial u^+}{\partial y^+}\right)_{x^+} + \left(\frac{\partial u^+}{\partial x^+}\right)_{y^+}\right\}\right]$$

After substitution of this and all other similar terms, an energy equation is derived as shown below; where the independent variables are x^+ and u^+

$$u^+ \frac{\partial\theta}{\partial x^+} + \pi u^{+2} \frac{\partial\theta}{\partial u^+} = \frac{\partial u^+}{\partial y^+} \frac{\partial}{\partial u^+}\left[\alpha^+ \left(\frac{\partial u^+}{\partial y^+}\right)\left(\frac{\partial\theta}{\partial u^+}\right)\right] \tag{45}$$

$$\alpha^+ = \frac{1}{Pr} + \frac{\epsilon_H}{\nu}$$

$$\pi = \frac{\delta\Delta}{\delta y^+}$$

and Δ is given by Equation (11c). Under conditions where π is a small number and is independent of u^+ and θ, the solution of this rather complex equation can be approached by a perturbation method. The variable u^+ is expressed by the expansion of Equation (12), already discussed. The dimensionless temperature, being a function of both u^+ and x^+, must be expressed in a double expansion of these two terms:

$$u^+ = u_0^+ + Zu_1^+ + Z^2u_2^+ + \ldots = \sum_{n=0}^{\infty} Z^n u_n^+ \tag{46a}$$

$$\theta = \theta_{0,0} + \pi\theta_{1,0} + Z\theta_{0,1} + Z\pi\theta_{1,1} + \pi^2\theta_{2,0} + \ldots$$

$$\theta = \sum_{i,j=0}^{i,j=\infty} \pi^i Z^j \theta_{i,j} \tag{46b}$$

Substituting these relations into the energy equation and collecting terms of like powers in π and Z gives a series of equations, the first few of which are

Zeroth Perturbation Equation

$$u_0^+ \frac{\partial\theta_{0,0}}{\partial x^+} = \frac{\partial u_0^+}{\partial y^+} \frac{\partial}{\partial u_0^+}\left[\alpha^+ \frac{\partial u_0^+}{\partial y^+} \frac{\partial\theta_{0,0}}{\partial u_0^+}\right] \tag{47}$$

First Perturbation Equations

$$u_0{}^+ \frac{\partial \theta_{1,0}}{\partial x^+} + u_0{}^+ \frac{\partial \theta_{0,0}}{\partial u_0{}^+} = \frac{\partial u_0{}^+}{\partial y^+} \frac{\partial}{\partial u_0{}^+}\left[\alpha^+ \frac{\partial u_0{}^+}{\partial y^+} \frac{\partial \theta_{0,0}}{\partial u_0{}^+}\right] \quad (48a)$$

$$u_0{}^+ \frac{\partial u_1{}^+}{\partial u_0{}^+} \frac{\partial \theta_{0,0}}{\partial x^+} + u_1{}^+ \frac{\partial \theta_{0,0}}{\partial x^+} + u_0{}^+ \frac{\partial \theta_{0,1}}{\partial x^+} =$$

$$= \frac{\partial u_0{}^+}{\partial y^+} \frac{\partial}{\partial u_0{}^+}\left[\alpha^+ \frac{\partial u_0{}^+}{\partial y^+}\left(\frac{\partial \theta_{0,1}}{\partial u_0{}^+} + \frac{\partial \theta_{0,0}}{\partial u_0{}^+}\right)\right] \quad (48b)$$

Consider now the zeroth-order term in the series for θ [Equation (46b)]. This is obtained from a solution of Equation (47). We have already shown that the zeroth-order term in the perturbation expansion for u^+ is given by Equation (16). Therefore

$$\frac{\delta u_0{}^+}{\partial y^+} = \frac{1}{1 + \epsilon/\nu}$$

Thus Equation (47) becomes

$$u_0{}^+ \frac{\partial \theta_{0,0}}{\partial x^+} = \frac{1}{1+\epsilon/\nu} \frac{\partial}{\partial u_0{}^+}\left[\frac{\alpha^+}{1+\epsilon/\nu} \frac{\partial \theta_{0,0}}{\partial u_0{}^+}\right] \quad (49)$$

This is, of course, identically the equation which Spalding and others (7, 8) have solved by approximate means to obtain temperature distributions and wall heat flux. The Spalding solution is thus the zeroth-order term in a small perturbation type of solution of the transformed energy equation.

The first term in Equation (46b), $\theta_{0,0}$, is a valid approximation of the true value of θ only (a) when the perturbation type of expansion is itself valid and (b) when successively higher terms in the expansion are small. It has been shown above that a perturbation type of solution is valid when Δ is small, a condition which was shown to exist at the wall. Even when this condition is satisfied, however, the first perturbation terms can be significant when the thermal boundary layer becomes large. Here the higher order terms obtained from the solution of Equations (48a) and (48b) introduce the effect of pressure gradient (through $u_1{}^+$) and of imperfect similarity.

It is possible to solve these equations for the first perturbation, $\theta_{0,1}$ and $\theta_{1,0}$, numerically and thus to develop a correction to Spalding's solutions analogous to the wake law correction for the law of the wall. However, the existence of similarity required to justify these refinements is not readily established.

The second paper in this series will present an alternative similarity concept which leads to a set of equations that can be solved numerically without the need for a zeroth-order perturbation type of approximation.

THE VELOCITY DEFECT LAW

This law suggests that the asymptotic approach of the velocity profile to the free-stream velocity is uniquely determined by the distance from the wall relative to the boundary layer thickness. Experimental data show that this approach is highly dependent upon the pressure gradient, but for a given pressure gradient it is independent of position along the body. An examination of the momentum equation written in these coordinates can shed some light on the conditions under which this type of similarity can be expected to be valid.

The velocity-defect variables are defined as

$$u_D = u_D(\xi, x) = \frac{u_\infty - u}{u_*} \quad (50)$$

$$F = \int_\infty^\xi u_D \, d\xi$$

Introducing these in the momentum equation, by a transformation of variables, produces the following equation:

$$\frac{\partial}{\partial \xi}\left[\frac{1+\epsilon/\nu}{1+\epsilon_c/\nu} \frac{\partial^2 F}{\partial \xi^2}\right] + a_1 \xi \frac{\partial^2 F}{\partial \xi^2} - a_2 \frac{\partial F}{\partial \xi} +$$

$$a_3\left(\frac{\partial F}{\partial \xi}\right) - a_4 F \frac{\partial^2 F}{\partial \xi^2} = a_5\left(\frac{\partial F}{\partial \xi} - \frac{u_\infty}{u_*}\right)^2 \frac{\partial \Delta_D}{\partial \xi} \quad (51)$$

where

$$a_1 = \frac{\delta}{\epsilon_c} \frac{d}{dx}(u_\infty \delta) \qquad a_4 = \frac{\delta}{\epsilon_c} \frac{d}{dx}(\delta u_*)$$

$$a_2 = \frac{\delta^2}{u_* \epsilon_c} \frac{d}{dx}(u_* u_\infty) \qquad a_5 = \frac{\delta^2 u_*}{\epsilon_c}$$

$$a_3 = \frac{\delta^2}{\epsilon_c} \frac{du_*}{dx} \qquad \Delta_D = \frac{(\partial F/\partial x)_\xi}{(\partial F/\partial \xi)_x - \dfrac{u_\infty}{u_*}} \quad (52)$$

Equation (51) contains no assumption other than the usual ones which are basic to the boundary layer equation itself.

In this form the equation clearly shows that similarity in $u_D - \xi$ coordinates exist only (a) when the coefficients a_1–a_4 are independent of x, or are small, (b) when Δ_D is a constant or varies linearly with ξ, and (c) when (ϵ/ν) is uniquely a function of ξ or is a constant. Only when all these conditions are satisfied is u_D a unique function of ξ. The pressure gradient term does not directly appear in this equation; however, it does make its influence felt through the coefficients, a_1–a_5.

All these conditions can be met only under very restrictive conditions. It was shown (see The Law of the Wake, above) that the eddy viscosity is not a unique function of ξ for values of ξ below about 0.16. For higher values, however, the eddy viscosity is essentially constant. This defect law, therefore, can not be expected to be valid below ξ of 0.16. In the region $\xi > 0.16$ we now examine the magnitude of the coefficients, a_1–a_4. Relative magnitudes can be estimated with the help of Equation (19) and the definition of Z to yield

$$\frac{a_2}{a_1} \cong -B\sqrt{C_f/2} << 1.0$$

$$\frac{a_3}{a_2} \cong B\sqrt{C_f/2}\, \frac{Z\, Re_\delta}{K_2 - Z\, Re_\delta \sqrt{2/C_f}} << 1.0$$

$$\frac{a_4}{a_3} \cong \sqrt{C_f/2}\left[1 + \frac{B\, K_2}{K_2 - Z\, Re_\delta\sqrt{2/C_f}}\right] << 1.0$$

At high Reynolds number where the drag coefficient is small, and if Δ_D is constant or a linear function of ξ, Equation (51) reduces to an ordinary differential equation of the form

$$\frac{d^2 u_D}{d\xi^2} + a_1\, \xi \frac{du_D}{d\xi} = 0 \quad (53)$$

Hinze (6) arrives at a similar equation although by somewhat different reasoning. He also showed that the solution of this equation could be expressed in error function form:

$$u_D = R\sqrt{\frac{\pi}{2a_1}}\left[1 - \text{erf}\left(\sqrt{\frac{a_1}{2}}\,\xi\right)\right] \quad (54)$$

where

$$R = \frac{u_* \delta}{\epsilon_c} = 13.5$$

By comparing his data with the data of Klebanoff (9) for zero pressure gradient, Hinze found that the value of a_1 was 3.54.

This constant must, of course, vary as pressure gradient changes, and a general relation can be found through the displacement distance.

$$\delta_D = \delta \int_0^{1.0} \frac{u_\infty - u}{u_\infty} d\xi \qquad (55)$$

Substituting for u_D in the integral, by means of Equation (54), gives

$$a_1 = 13.5 \frac{\delta}{\delta_D^*} \qquad (56)$$

where

$$\delta_D^* = \frac{\delta_D u_\infty}{u_*} \qquad (57)$$

Clauser (3) reports a value of $\delta_D^*/\delta = 3.6$ for zero pressure gradient. Using this value in Equation (56) gives a_1 equal to 3.5, which is in good agreement with Hinze's 3.54. Other values of a_1 for cases of positive and negative gradient can be obtained with Equation (56).

Equation (54) can be rearranged:

$$\left(\frac{\delta}{\delta_D^*} \right)^{1/2} u_D = 4.6 \left[1 - \text{erf} \left(2.6\, \xi \sqrt{\frac{\delta}{\delta_D^*}} \right) \right] \qquad (58)$$

This shows that, if the assumptions of similarity are valid, the defect law plotted in the coordinates $(u_D \sqrt{\delta/\delta_D^*})$ vs. $(\xi \sqrt{\delta/\delta_D^*})$ should be universal and independent of pressure gradient. The solution of Equation (58) appears in Figure 3 as a solid line. The data of Smith and Walker (11), Launder (10), and Clauser (3) are compared with the analytical solutions, and it is apparent that agreement is unsatisfactory over much of the boundary layer, not only over that region for $\xi < 0.16$. Thus the defect law must be considered a convenient means of plotting velocity distribution for a given pressure gradient, but in no sense does it represent a valid similarity type of solution to the boundary layer equations even for a restricted region.

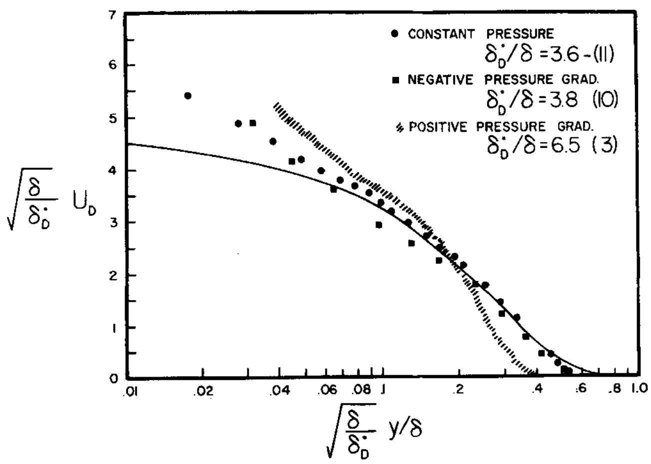

Fig. 3. Velocity defect law; comparison with data.

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NOTATION

a = constant in Equation (42)
$a_{1,2,3,4,5}$ = constants in Equations (51), (52)
A, B = constants in logarithmic wall profile [Equation (17)]
C_f = skin friction coefficient
F = $\int_\infty^\xi U_D d\xi$, Equation (50)
K_1 = coefficient of Coles' wake function
K_2 = Zp^+
K_3 = pressure gradient parameter, $Zp^+ y_m^+ = \frac{\delta}{u_*^2 \rho} \frac{dP}{dx}$
P = pressure
p^+ = dimensionless pressure parameter, Equation (11*b*)
T = temperature
T_∞ = free stream temperature
T_w = surface temperature
u = velocity component in x direction
u_0^+ = zeroth-order term in perturbation expansion equation for u^+ [Equation (12)]
u_1^+ = first-order term in perturbation expansion equation for u^+ [Equation (12)]
u_∞ = free stream velocity
u_* = friction velocity
u_D = dimensionless defect velocity, [Equation (50)]
u^+ = dimensionless velocity
v = velocity component in y direction
$W(\xi)$ = wake function
x = coordinate measuring distance along body
x^+ = dimensionless coordinate
y = coordinate measuring distance normal to body
y^+ = dimensionless coordinate
y_m^+ = y^+ at $y = \delta$
Z = perturbation parameter, Equation (11*a*)

Greek Letters

α = molecular thermal diffusivity
α^+ = dimensionless total thermal diffusivity
δ = boundary layer thickness
δ_D = displacement thickness
δ_D^* = displacement thickness parameter, Equation (57)
Δ = similarity parameter defined in Equation (11*b*)
ϵ = momentum eddy diffusivity
ϵ_H = thermal eddy diffusivity
ϵ^+ = dimensionless total momentum diffusivity
ρ = density
θ = dimensionless temperature
ν = kinematic viscosity
π = perturbation parameter, Equation (41)
ξ = y/δ dimensionless coordinate
ψ = stream function
ψ^+ = dimensionless stream function, Equation (9*b*)

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